JOURNAL OF APPROXIMATION THEORY **89**, 257–265 (1997)
ARTICLE NO. AT973041

Asymptotic Behavior of Orthogonal Polynomials Corresponding to a Measure with Infinite Discrete Part off a Curve

Rachid Benzine

Department of Mathematics, University of Annaba, B.P 12 El Hadjar, 23000 Annaba, Algeria

Communicated by W. Van Assche

Received January 4, 1996; accepted in revised form April 11, 1996

We determine the asymptotic behavior of orthogonal polynomials associated to a measure $\alpha = \beta + \gamma$, where β is a measure concentrated on a rectifiable Jordan curve and γ is an infinite discrete measure.

1. INTRODUCTION

Let α be a finite positive measure defined on the Borel sets of C. If $F =$ support α and if all moments of α exist, then the monic orthogonal polynomials associated to the measure α are given by

$$L_n(z) = z^n + \cdots, \qquad \int_F L_n(z)\, \bar{z}^p \, d\alpha = 0; \quad p = 0, 1, 2, ..., n-1. \tag{1.1}$$

If the measure α is not discrete, then for every n, the polynomial L_n exists and is unique.

In [2], we have studied the asymptotic behavior of the orthogonal polynomials $\{L_n\}$, where $F = E \cup \{z_k\}_{k=1}^{N}$, E a rectifiable Jordan curve and $z_k \in \Omega$, $\Omega = \text{exterior}(E)$; $\alpha = \beta + \gamma$; β is concentrated on E and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$ on the arc; $d\beta = \rho(\xi)\, |d\xi|$; and γ is a discrete measure with masses A_k at the points $z_k \in \text{Ext}(E)$, $k = 1, 2, ..., N$.

In this paper we generalize Theorem 4.1 of [2], when $F = E \cup \{z_k\}_{k=1}^{\infty}$; $\alpha = \beta + \gamma$; and E and β possess the same characteristics as in [2].

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}, \qquad A_k > 0, \qquad \sum_{k=1}^{\infty} A_k < +\infty.$$

0021-9045/97 $25.00

2. THE SPACE $H^2(\Omega, \rho)$

Suppose E is a rectifiable Jordan curve in the complex plane, $\Omega = \mathrm{Ext}(E)$, $G = \{w \in C / |w| > 1\}$ ($\infty \in \Omega$, $\infty \in G$), and $\Phi\colon \Omega \to G$ is the conformal mapping with $\Phi(\infty) = \infty$ and $\lim_{z \to \infty}(\Phi(z)/z) > 0$. Then

$$\Phi(z) = \frac{1}{C(E)} z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

in a neighborhood of infinity. We denote $\Psi = \Phi^{-1}$.

Let $H(\Omega)$ be the space of functions analytic in Ω. We say that $f \in H^2(\Omega)$ if $f \in H(\Omega)$ and

$$\int_{Er} |f(z)|^2 \, |dz| \leqslant C, \quad 1 < r \leqslant 2, \qquad Er = \{z\colon |\Phi(z)| = r\},$$

and C is a constant independent of r. A function $f \in H^2(\Omega)$ if and only if $f(\Psi(w)) \cdot \sqrt{\Psi'(w)} \in H^2(G)$, and a function $F \in H^2(G)$ if and only if $F(1/w') \in H^2(D)$; ($w' \in D$; $D = \{z \in C / |z| < 1\}$). The space $H^2(D)$ is well known (see [3, 5]).

Let $\rho(\xi)$ be an integrable nonnegative function on E. If the weight function $\rho(\xi)$ satisfies the Szegő condition

$$\int_E \log(\rho(\xi)) \, |\Phi'(\xi)| \, |d\xi| > -\infty, \tag{2.1}$$

then one can construct the so-called Szegő function $D_\rho(z)$ associated with domain Ω and weight function $\rho(\xi)$ with the following properties:

$$D_\rho \in H^2(\Omega); \quad D_\rho(z) \neq 0 \quad (z \in \Omega); \quad D_\rho(\infty) > 0; \quad |\tilde{D}_\rho(\xi)|^2 = \rho(\xi)$$

where $\tilde{D}_\rho(\xi) = \lim_{z \to \xi} D_\rho(z)$ (almost everywhere on E) (see [6]).

Let $f(z)$ be a function in $H(\Omega)$; we say that $f \in H^2(\Omega, \rho)$ if $(f \cdot D_\rho) \in H^2(\Omega)$. We find the principal properties of the space $H^2(\Omega, \rho)$ in the following theorem:

THEOREM 2.1 [2]. *If $f(z) \in H^2(\Omega, \rho)$ then almost everywhere on E the angular limit $\tilde{f}(\xi)$ exists: $\tilde{f}(\xi) = \lim_{z \to \xi} f(z)$. Furthermore,*

(1) $\tilde{f} \in L^2(E, \rho(\xi)\, |d\xi|)$

(2) $(H^2(\Omega, \rho), \|\cdot\|_\rho)$ *is a Hilbert space where*

$$\|f\|_\rho^2 = \langle f, f \rangle_\rho \qquad \text{and} \qquad \langle f, g \rangle_\rho = \int_E \tilde{f}(\xi)\, \bar{\tilde{g}}(\xi)\, \rho(\xi)\, |d\xi|,$$

for $f \in H^2(\Omega, \rho)$ and $g \in H^2(\Omega, \rho)$.

(3) *If $K \subset \Omega$, K compact, then there exists a constant $C(K)$ ($C(K)$ depends only on K) possessing the following property*:

$$\forall f \in H^2(\Omega, \rho), \quad \forall z \in K, \qquad |f(z)| \leqslant C(K) \, \|f\|_\rho. \tag{2.2}$$

3. ASYMPTOTIC BEHAVIOR

We now study the asymptotic behavior of orthogonal polynomials $\{L_n\}$ associated to the measure $\alpha = \beta + \gamma$. We say that a measure α belongs to the class S (denoted by $\alpha \in S$) if the absolutely continuous part β of α is such that:

$$d\beta = \rho(\xi) \, |d(\xi)|, \quad \rho\colon E \to R_+, \quad \int_E \rho(\xi) \, |d\xi| < +\infty,$$

and

$$\int_E \operatorname{Log}(\rho(\xi)) \, |\Phi'(\xi)| \, |d\xi| > -\infty. \tag{3.1}$$

The measure γ is such that

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}; \quad A_k > 0; \quad \sum_{k=1}^{\infty} A_k < +\infty. \tag{3.2}$$

We suppose that the moments of α exist.

Relations (1.1) become:

$$L_n(z) = z^n + \cdots; \qquad \int_E L_n(\xi) \, \bar{\xi}^p \rho(\xi) \, |d\xi| + \sum_{k=1}^{\infty} A_k L_n(z_k) \, \bar{z}_k^p = 0;$$
$$p = 0, 1, 2, ..., n-1. \tag{3.3}$$

DEFINITION 3.1. Let $\alpha = \beta + \gamma$ be such that $\alpha \in S$. We say that a measure α belongs to the class $\tilde{S}$ (denoted by $\alpha \in \tilde{S}$) if the discrete part γ of α satisfies (3.2), and moreover:

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1 \right) < \infty \tag{3.4}$$

$$\frac{\int_E |L_n(\xi)|^2 \, \rho(\xi) \, |d\xi|}{\sum_{k=1}^{\infty} A_k \, |L_n(z_k)|^2} \geqslant \frac{1}{(\prod_{k=1}^{\infty} |\Phi(z_k)|)^2 - 1}. \tag{3.5}$$

The extremal polynomial L_n is such that

$$\|L_n\|_\alpha^2 = \min\{\|Q_n\|_\alpha^2 : Q_n = z^n + a_{n-1}z^{n-1} + \cdots + a_0\} = m_n(\alpha), \quad (3.6)$$

where

$$\|L_n\|_\alpha^2 = \int_E |L_n(\xi)|^2 \rho(\xi)\, |d\xi| + \sum_{k=1}^{\infty} A_k\, |L_n(z_k)|^2.$$

Define $\mu_n(\alpha)$, $\mu(\alpha)$, and $\hat{\mu}(\alpha)$ as the extremal values of the following problems:

$$\mu_n(\alpha) = \min\left\{\int_E |\varphi_n(\xi)|^2 \rho(\xi)\, |d\xi| + \sum_{k=1}^{\infty} A_k\, |\Phi(z_k)|^{2n}\, |\varphi_n(z_k)|^2 : \right.$$

$$\left. \varphi_n = \frac{Q_n}{[C(E)\,\Phi]^n};\ Q_n \in P_n;\ \varphi_n(\infty) = 1 \right\} \quad (3.7)$$

(P_n is the set of polynomials of degree at most n.)

$$\mu(\alpha) = \inf\{\|\varphi\|_\rho^2 : \varphi \in H^2(\Omega, \rho),\ \varphi(\infty) = 1\} \quad (3.8)$$

$$\hat{\mu}(\alpha) = \inf\{\|\varphi\|_\rho^2 : \varphi \in H^2(\Omega, \rho);\ \varphi(\infty) = 1;\ \varphi(z_k) = 0,\ k = 1, 2, \ldots\} \quad (3.9)$$

We denote respectively by φ_n^*, φ^*, and $\hat{\varphi}^*$ the extremal functions of the problems (3.7), (3.8), and (3.9). We have

$$\varphi_n^* = \frac{L_n}{[C(E)\,\Phi]^n} \qquad \text{and} \qquad \mu_n(\alpha) = \frac{m_n(\alpha)}{C(E)^{2n}}. \quad (3.10)$$

We denote by Φ_n the polynomial part of the Laurent expansion of Φ^n in a neighborhood of infinity and $\lambda_n = \Phi^n - \Phi_n$.

Geronimus introduced in [1] a class of curves denoted by τ, and defined it as follows:

A rectifiable Jordan curve E belongs to the class τ *if*

$\lambda_n(\xi) \to 0$, *uniformly on E.* (3.11)

We find in [1, pp. 22, 23] the following families of curves belonging to the class τ:

(a) *The analytic curves.* The property (3.11) has been proved by Faber;

$$|\lambda_n(\xi)| \leqslant Cr^n, \qquad 0 < r < 1$$

(see Szegő [7]).

(b) *The smooth curves* $z(t)$, whose derivatives $z'(t)$ satisfy the Lipschitz condition:

$$|z'(t_1) - z'(t_2)| < L\,|t_1 - t_2|^{\alpha}, \qquad 0 < \alpha \leqslant 1.$$

In this case, we get from Korovkin [4] the following inequality: $|\lambda_n(\xi)| < C/n^{\alpha_1}$, $0 < \alpha_1 < \alpha$, $\xi \in E$.

(c) We can find in [1] other families of curves satisfying (3.11) whose descriptions are not as explicit as the former ones.

The result of this paper is

THEOREM 3.1. *Let* $\alpha = \beta + \gamma$ *such that* $\alpha \in \tilde{S}$, $E \in \tau$, *and* $\{L_n\}$ *is the system of monic orthogonal polynomials associated to* α. *Then we have*:

(1) $\lim_{n \to \infty} \|(L_n/[C(E)\,\Phi]^n) - \hat{\varphi}^*\|_{\rho}^2 = 0$ (3.12)

(2) $L_n(z) = C(E)^n\,\Phi^n(z)[(\varphi^*(z) \cdot B(z)) + \varepsilon_n(z)]$, (3.13)

$$B(z) = \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{(\Phi(z) \cdot \overline{\Phi(z_k)}) - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)}, \tag{3.14}$$

$\varepsilon_n \to 0$, *uniformly on compact sets of* Ω.

We begin by proving the next lemmas:

LEMMA 3.1. *Let* $\varphi \in H^2(\Omega, \rho)$ *such that* $\varphi(\infty) = 1$ *and* $\varphi(z_k) = 0$, $k = 1, 2, \ldots$ *and* B *the Blaschke product* (3.14). *Then*:

(1) $B \in H^2(\Omega, \rho)$; $B(\infty) = 1$; $|\tilde{B}(\xi)| = \prod_{k=1}^{\infty} |\Phi(z_k)|$ $(\tilde{B}(\xi) = \lim_{z \to \xi} B(z))$

(2) $\dfrac{\varphi}{B} \in H^2(\Omega, \rho)$ *and* $\left(\dfrac{\varphi}{B}\right)(\infty) = 1.$

Proof of Lemma 3.1. (1) is evident if we remark that

$$B(z) = k_1(\Phi(z)),$$

where

$$k_1(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w\bar{w}_k - 1} \cdot \frac{|w_k|^2}{w_k}, \qquad w_k = \Phi(z_k), \quad |w| > 1$$

and

$$\forall w\colon k_1(w) = k_2(w) \cdot \prod_{k=1}^{\infty} |w_k|,$$

where

$$k_2(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w\bar{w}_k - 1} \frac{|w_k|}{w_k},$$

k_2 is bounded in G, and $|\widetilde{k}_2(e^{i\theta})| = 1$ almost everywhere. $\widetilde{k}_2$ is the angular limit of k_2 (see [3] and [5]).

(2) $B(z_k) = 0$, $k = 1, 2, \ldots$, and $B(z) \neq 0$ if $z \neq z_k$. Then $\{z_k\}$ are regular for φ/B. Therefore φ/B has an analytic extension in Ω.

Now we show that $\varphi/B \in H^2(\Omega, \rho)$. It suffices to prove that

$$L(w') \in H^2(D),$$

where $D = \{w' \colon |w'| < 1\}$ and

$$L(w') = \frac{\varphi(\Psi(1/w'))\, D_\rho(\Psi(1/w')) \cdot \sqrt{\Psi'(1/w')}}{B(\Psi(1/w'))}.$$

Effectively, we have that

$$F(w') = \varphi\left(\Psi\left(\frac{1}{w'}\right)\right) \cdot D_\rho\left(\Psi\left(\frac{1}{w'}\right)\right) \cdot \sqrt{\Psi'\left(\frac{1}{w'}\right)} \in H^2(D),$$

and

$$F(w'_k) = 0 \qquad \left(w'_k = \frac{1}{\Phi(z_k)},\ k = 1, 2, \ldots\right).$$

Now if we consider the classical Blaschke product $K(w')$ [3, 5]:

$$K(w') = \prod_{k=1}^{\infty} \frac{w' - w'_k}{w'\bar{w}'_k - 1} \cdot \frac{|w'_k|}{w'_k}, \qquad |w'| < 1,$$

and

$$B_1(w') = B\left(\Psi\left(\frac{1}{w'}\right)\right),$$

then we have

$$B_1(w') = K(w') \cdot \prod_{k=1}^{\infty} \frac{1}{|w'_k|}, \qquad |w'| < 1.$$

Finally,

$$L(w')=\frac{F(w')}{K(w')}\cdot\frac{1}{\prod_{k=1}^{\infty}(1/|w'_k|)}\qquad\text{and}\qquad\frac{F}{K}\in H^2(D)\quad([3,5]).$$

LEMMA 3.2. *The extremal functions $\hat{\varphi}^*$ and φ^* are connected by*

$$\hat{\varphi}^*(z)=B(z)\cdot\varphi^*(z)\qquad\textit{and}\qquad\hat{\mu}(\alpha)=\mu(\alpha)\cdot\left(\prod_{k=1}^{\infty}|\Phi(z_k)|\right)^2.$$

Proof of Lemma 3.2. The proof is the same as that of Lemma 4.2 of [2]. We replace the finite Blaschke product by the infinite product B.

Proof of Theorem 3.1. As in Theorem 4.1 of [2], the essential step is to prove

$$\lim\mu_n(\alpha)=\hat{\mu}(\alpha).\tag{3.15}$$

(A) An upper bound of $\overline{\lim}\,\mu_n(\alpha)$. We have

$$\mu_n(\alpha)=\frac{1}{C(E)^{2n}}\left(\int_E|L_n(\xi)|^2\,\rho(\xi)\,|d\xi|+\sum_{k=1}^{\infty}A_k\,|L_n(z_k)|^2\right)$$

and

$$\begin{aligned}\min&\left\{\int_E|\varphi_n(\xi)|^2\,\rho(\xi)\,|d\xi|+\sum_{k=1}^{\infty}A_k\,|\Phi(z_k)|^{2n}\,|\varphi_n(z_k)|^2:\right.\\&\qquad\left.\varphi_n=\frac{Q_nB}{[C(E)\,\Phi]^n};\ \varphi_n(\infty)=1\right\}\\&=\frac{1}{C(E)^{2n}}\left(\left(\prod_{k=1}^{\infty}|\Phi(z_k)|\right)^2\cdot\int_E|L_n(\xi)|^2\,\rho(\xi)\,|d\xi|\right)=\mu_n(\rho')\end{aligned}$$

$$\rho'=\rho\cdot\left(\prod_{k=1}^{\infty}|\Phi(z_k)|\right)^2$$

and

$$\mu_n(\rho')=\min\left\{\int_E|\varphi_n(\xi)|^2\,\rho'(\xi)\,|d\xi|:\varphi_n=\frac{Q_n}{[C(E)\,\Phi]^n};\ \varphi_n(\infty)=1\right\},$$

$$\alpha\in\tilde{S}\Rightarrow\mu_n(\alpha)\leqslant\mu_n(\rho')\qquad\text{and}\qquad\mu_n(\rho')\to\mu(\rho')$$

(see [2]), where

$$\mu(\rho') = \inf\left\{\int_E |\tilde{\varphi}(\xi)|^2 \rho'(\xi)\, |d\xi| : \varphi \in H^2(\Omega, \rho);\ \varphi(\infty) = 1\right\}.$$

Finally, we have

$$\overline{\lim}\, \mu_n(\alpha) \leqslant \mu(\rho'). \tag{3.16}$$

We prove without difficulty as in Lemma 4.1 of [2] (we replace ω_ℓ in [2] by B) that

$$\mu(\rho') = \hat{\mu}(\alpha). \tag{3.17}$$

$$(3.16) \text{ and } (3.17) \quad \Rightarrow \quad \overline{\lim}\, \mu_n(\alpha) \leqslant \hat{\mu}(\alpha). \tag{3.18}$$

(B) A lower bound of $\underline{\lim}\, \mu_n(\alpha)$. We have

$$\forall \ell > 0, \qquad \mu_n(\alpha) \geqslant \mu_n(\ell), \tag{3.19}$$

where

$$\mu_n(\ell) = \frac{1}{C(E)^{2n}} \min\left\{\int_E |Q(\xi)|^2 \rho(\xi)\, |d\xi| + \sum_{k=1}^{\ell} A_k\, |Q(z_k)|^2 : Q(z) = z^n + \cdots + a_0\right\}.$$

It is well known (from [2]) that $\mu_n(\ell) \xrightarrow[n \to +\infty]{} \mu(\ell)$, where

$$\mu(\ell) = \inf\{\|\varphi\|_\rho^2 : \varphi \in H^2(\Omega, \rho);\ \varphi(\infty) = 1;\ \varphi(z_k) = 0,\ k = 1, 2, ..., \ell\}.$$

We have also that

$$\mu(\ell) = \mu(\alpha) \cdot \left(\prod_{k=1}^{\ell} |\Phi(z_k)|\right)^2$$

and

$$\hat{\mu}(\alpha) = \mu(\alpha) \cdot \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right)^2 \qquad \text{(Lemma 3.2)}.$$

We then obtain that

$$\forall \ell > 0, \qquad \underline{\lim}\, \mu_n(\alpha) \geqslant \mu(\alpha) \cdot \left(\prod_{k=1}^{\ell} |\Phi(z_k)|\right)^2,$$

and finally that

$$\underline{\lim}\ \mu_n(\alpha) \geqslant \mu(\alpha) \cdot \left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right)^2 = \hat{\mu}(\alpha).$$

$$(\text{A}) \text{ and } (\text{B}) \quad \Rightarrow \quad \lim \mu_n(\alpha) = \hat{\mu}(\alpha). \tag{3.20}$$

We obtain (3.12) of Theorem 3.1 by proceeding as in [2, pp. 42, 43]. Then (3.12), (2.2), and Lemma 3.2 imply (3.13) of Theorem 3.1.

REFERENCES

1. J. Geronimus, Extremal problems in the space $L^2(\sigma)$, *Mat. Sb.* **31** (1952), 3–26 [in Russian].
2. V. Kaliaguine and R. Benzine, Sur la formule asymptôtique des polynomes orthogonaux associés à une mesure concentrée sur un contour plus une partie discrète finie, *Bull. Soc. Math. Belg. Ser. B* **41**, No. 1 (1989), 29–46.
3. P. Koossis, "Introduction to H^p Spaces," London Math. Soc. Lecture Notes Series, Vol. 40, Cambridge Univ. Press, Cambridge, 1980.
4. P. P. Korovkin, On polynomials, orthogonal on a rectifiable curve, with respect to a weight function, *Mat. Sb.* **9** (1941), 469–484 [in Russian].
5. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1968.
6. V. I. Smirnov and N. A. Lebedev, "The Constructive Theory of Functions of a Complex Variable," Nauka, Moscow, 1964 [in Russian]; M.I.T. Press, Cambridge, MA, 1968 [Engl. transl.].
7. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th ed., American Math. Society, Providence, RI, 1975.

Printed in Belgium